Markscheme

May 2017

Further mathematics

Higher level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2017**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working*.

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \ \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $H_0 : \mu = 5.5; H_1 : \mu \neq 5.5$ ***A1***

[1 mark]

(b) $\sum x = 53.9,\ \hat{\mu} = 5.39$ ***(M1)A1***

$\sum x^2 = 290.7132,\ \hat{\sigma}^2 = 0.0214$ ***(M1)A1***

Note: Accept any answer that rounds correctly to 0.021.

[4 marks]

(c) (i) attempt to use the t-test ***(M1)***

$t = -2.38$ (Accept $+2.38$) ***(A1)***

$\text{DF} = 9$ ***(A1)***

p-value $= 0.0412$ ***A1***

(ii) the claim is not supported (not accepted, rejected) at the 5% level of significance ***A1***

[5 marks]

Total [10 marks]

2. (a) multiplying both sides by a^{p-2}, ***M1***

$a^{p-1}x \equiv a^{p-2}b(\text{mod } p)$ ***A1***

using $a^{p-1} \equiv 1(\text{mod } p)$ ***R1***

therefore, $x \equiv a^{p-2}b(\text{mod } p)$ ***AG***

[3 marks]

(b) using the above result,

$x \equiv 3^{17} \times 13(\text{mod}19)\left(\equiv 16\,7882\,2119(\text{mod}19)\right)$ ***A1***

$\equiv 17(\text{mod}19)$ ***(M1)A1***

$x = 112$ ***A1***

[4 marks]

Total [7 marks]

3. (a) using row operations on 4×5 matrix, ***M1***

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -9 & 3 & -15 \\ 0 & -3 & 1 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ \lambda-10 \\ \mu-6 \end{bmatrix} \begin{matrix} \\ \text{row2} - 2\times\text{row1} \\ \text{row3} - 5\times\text{row1} \\ \text{row4} - 3\times\text{row1} \end{matrix}$$ ***A2***

or any alternative correct row reductions

Note: Award ***A1*** for two correct row reductions.

$\lambda = 7$ ***A1***

$\mu = 5$ ***A1***

[5 marks]

(b) let $x_3 = \alpha$, $x_4 = \beta$ ***M1***

$x_2 = \dfrac{1 + \alpha - 5\beta}{3}$ ***A1***

$x_1 = \dfrac{4 - 5\alpha + \beta}{3}$ ***A1***

Note: Alternative solutions are available.

[3 marks]

(c) the rank is 2 ***A1***

because the matrix has 2 independent rows or a correct comment based on the use of rref ***R1***

[2 marks]

Total [10 marks]

4. (a) let M, F denote the weights of the male, female

consider $D = M - 2F$ ***(M1)***

$\mathrm{E}(D) = 80 - 2 \times 54 = -28$ ***A1***

$\mathrm{Var}(D) = 7^2 + 4 \times 5^2$ ***(M1)***

$= 149$ ***A1***

$\mathrm{P}(M > 2F) = \mathrm{P}(D > 0)$ ***(M1)***

$= 0.0109$ ***A1***

Note: Accept any answer that rounds correctly to 0.011.

[6 marks]

continued…

Question 4 continued

(b) consider S = $\sum_{i=1}^{3} M_i + \sum_{i=1}^{6} F_i$ ***(M1)***

Note: Condone the use of the incorrect notation $3M + 6F$.

$\text{E}(S) = 3 \times 80 + 6 \times 54 = 564$ ***A1***

$\text{Var}(S) = 3 \times 7^2 + 6 \times 5^2$ ***(M1)***

$= 297$ ***A1***

$\text{P}(S > 550) = 0.792$ ***A1***

Note: Accept any answer that rounds correctly to 0.792.

[5 marks]

Total [11 marks]

5. (a) since $\sum u_n$ is convergent, it follows that $\lim_{n\to\infty} u_n = 0$ ***R1***

therefore, there exists N such that for $n \geq N$, $u_n < 1$ ***R1***

Note: Accept as n gets larger, eventually $u_n < 1$.

therefore (for $n \geq N$), $u_n^2 < u_n$ ***R1***

by the comparison test, $\sum u_n^2$ is convergent ***R1***

[4 marks]

(b) (i) the converse proposition is that if $\sum u_n^2$ is convergent, then $\sum u_n$ is also convergent ***A1***

(ii) a suitable counter-example is

$u_n = \frac{1}{n}\left(\text{for which } \sum u_n^2 \text{ is convergent but } \sum u_n \text{ is not convergent}\right)$ ***A1***

[2 marks]

Total [6 marks]

6. (a) the order is 6 ***A1***

tracking 1 through successive powers of P returns to 1 after 6 transitions (or equivalent) ***R1***

[2 marks]

(b) $P^2 = (1\ 5\ 4)(2\ 6\ 3)$ or $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{pmatrix}$ ***(M1)A1***

[2 marks]

continued…

Question 6 continued

(c) since P is of order 6, P^3 will be of order 2 ***R1***

$P^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$ ***(M1)(A1)***

$P^3 =$ (1 2)(3 4)(5 6) ***A1***

[4 marks]

Total [8 marks]

7. (a) (i) $f'(x) = \dfrac{e^x - e^{-x} - 2\sin x}{4}$ ***(A1)***

$f''(x) = \dfrac{e^x + e^{-x} - 2\cos x}{4}$ ***(A1)***

$f'''(x) = \dfrac{e^x - e^{-x} + 2\sin x}{4}$ ***(A1)***

$f^{(4)}(x) = \dfrac{e^x + e^{-x} + 2\cos x}{4} = f(x)$ ***AG***

(ii) therefore,

$f(0) = 1$ and $f^{(4)}(0) = 1$ ***(A1)***

$f'(0) = f''(0) = f'''(0) = 0$ ***(A1)***

the sequence of derivatives repeats itself so the next non-zero derivative is $f^{(8)}(0) = 1$ ***(A1)***

the MacLaurin series is $1 + \dfrac{x^4}{4!} + \dfrac{x^8}{8!}(+\ldots)$ ***(M1)A1***

[8 marks]

(b) (i) $p = P(X = 0) + P(X = 4) + P(X = 8) + \ldots$ ***(M1)***

$= \dfrac{e^{-\mu}\mu^0}{0!} + \dfrac{e^{-\mu}\mu^4}{4!} + \dfrac{e^{-\mu}\mu^8}{8!} + \ldots$ ***A1***

(ii) $p = e^{-\mu}\left(1 + \dfrac{\mu^4}{4!} + \dfrac{\mu^8}{8!} + \ldots\right)$ ***A1***

$= e^{-\mu} f(\mu)$ ***AG***

(iii) $p = e^{-3}\left(\dfrac{e^3 + e^{-3} + 2\cos 3}{4}\right)$ ***(M1)***

$= 0.226$ ***A1***

[5 marks]

Total [13 marks]

8. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$ ***(M1)***

$= \frac{2a}{2at} = \frac{1}{t}$ ***A1***

the gradient of the normal $= -t$ ***(A1)***

the equation of the normal at T is $y - 2at = -t\left(x - at^2\right)$ ***A1***

substituting the coordinates of S, ***M1***

$2as - 2at = -t\left(as^2 - at^2\right)$

$2a(s - t) = -at(s - t)(s + t)$ ***A1***

$2 = -t(s + t) = -st - t^2$ ***A1***

$t^2 + st + 2 = 0$ ***AG***

[7 marks]

(b) gradient of OT $= \frac{2at}{at^2} = \frac{2}{t}$ ***A1***

gradient of OS $= \frac{2as}{as^2} = \frac{2}{s}$ ***(A1)***

the condition for perpendicularity is $\frac{2}{t} \times \frac{2}{s} = -1$ ***M1***

$t^2 - 4 + 2 = 0$ ***A1***

$t = \pm\sqrt{2}$ ***A1***

[5 marks]

Total [12 marks]

9. (a) consider $\lim\limits_{x \to \infty} \frac{x^n}{\mathrm{e}^x}$ ***M1***

its value is $\frac{\infty}{\infty}$ so we use l’Hôpital’s rule

$\lim\limits_{x \to \infty} \frac{x^n}{\mathrm{e}^x} = \lim\limits_{x \to \infty} \frac{nx^{n-1}}{\mathrm{e}^x}$ ***(A1)***

its value is still $\frac{\infty}{\infty}$ so we need to differentiate numerator and denominator a further n–1 times ***(R1)***

this gives $\lim\limits_{x \to \infty} \frac{n!}{\mathrm{e}^x}$ ***A1***

since the numerator is finite and the denominator $\to \infty$,
the limit is zero ***AG***

[4 marks]

continued…

Question 9 continued

(b) (i) attempt at integration by parts $\left(I_n = -\int_1^\infty x^n \mathrm{d}\left(\mathrm{e}^{-x}\right)\right)$ ***M1***

$I_n = -\left[x^n \mathrm{e}^{-x}\right]_1^\infty + n\int_1^\infty x^{n-1}\mathrm{e}^{-x}\mathrm{d}x$ ***A1A1***

$= \mathrm{e}^{-1} + nI_{n-1}$ ***A1***

$\alpha = \beta = 1$

(ii) $I_3 = \mathrm{e}^{-1} + 3I_2$ ***M1***

$= \mathrm{e}^{-1} + 3\left(\mathrm{e}^{-1} + 2I_1\right)$ ***A1***

$= 4\mathrm{e}^{-1} + 6\left(\mathrm{e}^{-1} + I_0\right)$ ***A1***

$= 4\mathrm{e}^{-1} + 6\mathrm{e}^{-1} + 6\int_1^\infty \mathrm{e}^{-x}\mathrm{d}x$

$= 10\mathrm{e}^{-1} - 6\left[\mathrm{e}^{-x}\right]_1^\infty$ ***A1***

$= 16\mathrm{e}^{-1}$ ***A1***

[9 marks]

Total [13 marks]

10. (a) closure: let $\boldsymbol{A}, \boldsymbol{B} \in G$

(because $\boldsymbol{AB}$ is a 2×2 matrix)

and $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) = 1 \times 1 = 1$ ***M1A1***

identity: the 2×2 identity matrix has determinant 1 ***R1***

inverse: let $\boldsymbol{A} \in G$. Then $\boldsymbol{A}$ has an inverse because it is non-singular ***(R1)***

since $\boldsymbol{AA}^{-1} = \boldsymbol{I}$, $\det(\boldsymbol{A})\det\left(\boldsymbol{A}^{-1}\right) = \det(\boldsymbol{I}) = 1$ therefore $\boldsymbol{A}^{-1} \in G$ ***R1***

associativity is assumed

the four axioms are satisfied therefore $\{G, *\}$ is a group ***AG***

[5 marks]

(b) closure: let $\boldsymbol{A}, \boldsymbol{B} \in H$. Then $\boldsymbol{AB} \in H$ because the arithmetic involved produces elements that are integers ***R1***

inverse: $\boldsymbol{A}^{-1} \in H$ because the calculation of the inverse involves interchanging the elements and dividing by the determinant which is 1 ***R1***

the identity (and associativity) follow as above ***R1***

therefore $\{H, *\}$ is a subgroup of $\{G, *\}$ ***A1***

Note: Award the ***A1*** only if the first two ***R1*** marks are awarded but not necessarily the third ***R1***.

Note: Accept subgroup test.

[4 marks]

Total [9 marks]

11. (a) (i) the sum of degrees of the vertices is even (36) or the sum of degrees of the vertices is twice the number of edges ***A1***

(ii) the number of edges (e) is 18 ***A1***

using Euler's relation $v - e + f = 2$ ***M1***

$f = 2 + 18 - 8 = 12$ ***A1***

[4 marks]

(b) if K is planar then $e \le 3v - 6$ ***M1***

$v = 8$ and $e = 19$ ***A1***

the inequality is not satisfied so K is not planar ***A1AG***

[3 marks]

(c) (i) let PQRP be a cycle of length 3 in a graph ***M1***

Note: Accept a diagram instead of this statement.

suppose the graph is bipartite

then P must belong to one of the two disjoint sets of vertices and Q, R must belong to the other disjoint set ***R1***

but Q, R cannot belong to the same set because they are linked ***R1***

therefore the graph cannot be bipartite ***AG***

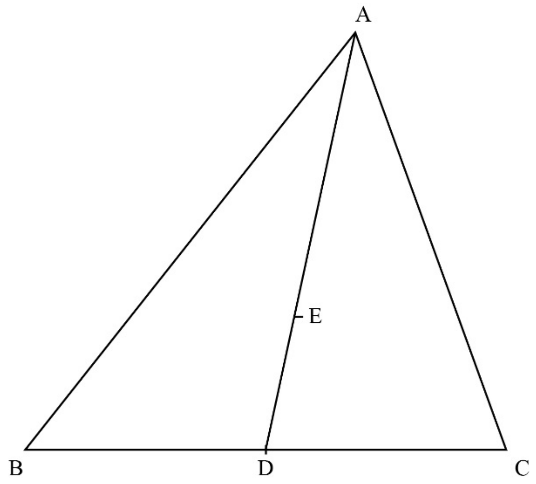
(ii) for example, a suitable cycle of order 3 is AFHA ***(M1)A1***

Note: Award ***M1*** for a valid attempt at drawing the complement or constructing its adjacency table.

[5 marks]

Total [12 marks]

12. (a) (i)



$$\boldsymbol{d} = \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$$ ***(M1)A1***

$$\boldsymbol{e} = \boldsymbol{d} + \frac{1}{3}(\boldsymbol{a} - \boldsymbol{d})$$ ***M1***

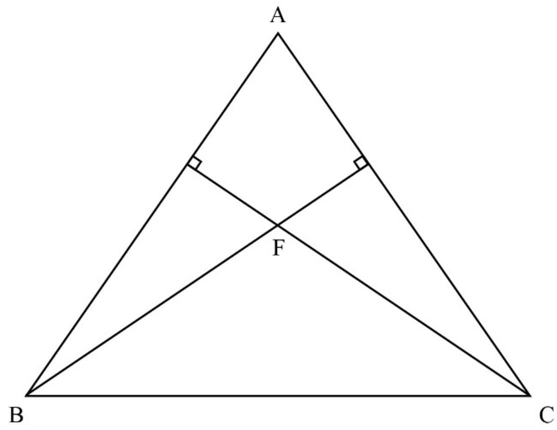
$$= \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) + \frac{1}{3}\left(\boldsymbol{a} - \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})\right)$$ ***A1***

$$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$$ ***AG***

(ii) (because of the symmetry of the result), the other two medians also pass through E. ***R1***

[5 marks]

(b) (i)



$\overrightarrow{\mathrm{BF}} = \boldsymbol{f} - \boldsymbol{b}$ and $\overrightarrow{\mathrm{AC}} = \boldsymbol{c} - \boldsymbol{a}$ ***A1***

since FB is perpendicular to AC, $(\boldsymbol{b} - \boldsymbol{f}) \cdot (\boldsymbol{c} - \boldsymbol{a}) = 0$ ***R1AG***

similarly since FC is perpendicular to BA, $(\boldsymbol{c} - \boldsymbol{f}) \cdot (\boldsymbol{a} - \boldsymbol{b}) = 0$ ***R1AG***

continued…

Question 12 continued

(ii) expanding these equations and adding, ***M1***

$\boldsymbol{b}\cdot\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{f}\cdot\boldsymbol{c} + \boldsymbol{f}\cdot\boldsymbol{a} + \boldsymbol{c}\cdot\boldsymbol{a} - \boldsymbol{c}\cdot\boldsymbol{b} - \boldsymbol{f}\cdot\boldsymbol{a} + \boldsymbol{f}\cdot\boldsymbol{b} = 0$ ***A1***

$-\boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{f}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a} + \boldsymbol{f}\cdot\boldsymbol{b} = 0$ ***A1***

leading to $(\boldsymbol{a} - \boldsymbol{f})\cdot(\boldsymbol{c} - \boldsymbol{b}) = 0$ ***AG***

(iii) this result shows that AF is perpendicular to BC so that the three altitudes are concurrent (at F) ***R1***

[7 marks]

Total [12 marks]

13. (a) (i) $\dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ is $\mathrm{N}(0, 1)$ or it has the Z-distribution ***A1***

(ii) attempt to make a probability statement ***R1***

therefore with probability 0.95,

$-1.96 \le \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \le 1.96$ ***A1***

$-1.96\dfrac{\sigma}{\sqrt{n}} \le \bar{X} - \mu \le 1.96\dfrac{\sigma}{\sqrt{n}}$ ***A1***

$1.96\dfrac{\sigma}{\sqrt{n}} \ge \mu - \bar{X} \ge -1.96\dfrac{\sigma}{\sqrt{n}}$ ***A1***

$\bar{X} + 1.96\dfrac{\sigma}{\sqrt{n}} \ge \mu \ge \bar{X} - 1.96\dfrac{\sigma}{\sqrt{n}}$

Note: Award the final ***A1*** for either of the above two lines.

$\bar{X} - 1.96\dfrac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\dfrac{\sigma}{\sqrt{n}}$ ***AG***

[5 marks]

(b) (i) you cannot make a probability statement about a constant lying in a constant interval **OR** the mean either lies in the interval or it does not ***A1***

(ii) the confidence interval is the observed value of a random interval **OR** if the process is carried out a large number of times, μ will lie in the interval 95% of the times ***A1***

[2 marks]

Total [7 marks]

14. (a) using the binomial theorem,

$$10^n = (11-1)^n = 11^n - n\times 11^{n-1} + \frac{n(n-1)}{2}\times 11^{n-2} + \ldots + n\times 11\times(-1)^{n-1} + (-1)^n$$

M1A1

since every term except the last one is divisible by 11, it follows that ***R1***

$10^n \equiv (-1)^n (\text{mod}\,11)$ ***AG***

[3 marks]

(b) (i) consider the decimal number

$N = a_n a_{n-1} \ldots a_1 a_0$ where n is odd (*ie*, an even number of digits) ***R1***

so $N = a_n \times 10^n + a_{n-1}\times 10^{n-1} + \ldots + a_1 \times 10 + a_0$ ***A1***

using the result in (a), since n is odd,

$N \equiv a_n \times(-1) + a_{n-1}\times(+1) + \ldots + a_1\times(-1) + a_0 (\text{mod}\,11)$ ***A1***

since N is palindromic,

$a_n = a_0\,;\ a_{n-1} = a_1;\ \ldots$

therefore,

$N \equiv (a_0 - a_n) + (a_{n-1} - a_1) + \ldots$ ***M1***

$\equiv 0\ (\text{mod}\ 11)$ ***A1***

hence N is divisible by 11 ***AG***

(ii) for example, 131 (or even 3) is not divisible by 11 ***A1***

[6 marks]

Total [9 marks]

15. (a) (i) let $\alpha_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 = \boldsymbol{0}$

take the dot product with $\boldsymbol{v}_1$ ***M1***

$\alpha_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 \cdot \boldsymbol{v}_1 + \alpha_3 \boldsymbol{v}_3 \cdot \boldsymbol{v}_1 = \boldsymbol{0}$ ***A1***

because the vectors are orthogonal, $\boldsymbol{v}_2 \cdot \boldsymbol{v}_1 = \boldsymbol{v}_3 \cdot \boldsymbol{v}_1 = \boldsymbol{0}$ ***R1***

and since $\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 > 0$ it follows that $\alpha_1 = 0$ and similarly, $a_2 = \alpha_3 = 0$ ***R1***

so $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 = \boldsymbol{0} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ therefore $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent ***R1AG***

(ii) the three vectors form a basis for $\mathbb{R}^3$ because they are (linearly) independent ***R1***

[6 marks]

(b) (i) $\begin{bmatrix}1\\0\\1\end{bmatrix}.\begin{bmatrix}-1\\1\\1\end{bmatrix} = 0\ ;\ \begin{bmatrix}1\\0\\1\end{bmatrix}.\begin{bmatrix}1\\2\\-1\end{bmatrix} = 0\ ;\ \begin{bmatrix}-1\\1\\1\end{bmatrix}.\begin{bmatrix}1\\2\\-1\end{bmatrix} = 0$ ***M1A1***

therefore the vectors form an orthogonal basis ***AG***

continued…

Question 15 continued

(ii) let $\begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \nu \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ***M1***

$\lambda - \mu + \nu = 2$

$\mu + 2\nu = 8$ ***A1***

$\lambda + \mu - \nu = 0$

the solution is

$\begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\left(\begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right)$ ***A1***

[5 marks]

Total [11 marks]